Poly symmetric polyhedra

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Definition

Axis of symmetry

A finite polyhedron is said to be symmetric with respect to this straight line if it makes one revolution around a straight line through it and overlaps its original position in the center at least once. This straight line is called the axis of symmetry.

If the polyhedron overlaps its original position n times during one rotation, the polyhedron is said to be n times symmetric with respect to its axis, and this axis is called the n times axis. Especially when n = 2, it is said to be simply symmetric. Below, the axes are represented by Xm, Yn..., where n is called the number of times.

Poly symmetric polyhedron

A polyhedron having two or more axes of symmetry is called a poly symmetric polyhedron.

When a poly symmetric polyhedron is rotated around one axis Xn, this polyhedron overlaps in-situ every 1 / n rotation, so any other axis Y is sequentially Y', Y "... overlapping. At this time, Y, Y', Y "... are said to be in corresponding positions around Xn. Also, at this time, one face or vertex of the polyhedron also sequentially overlaps with any of the other faces or vertices, so that the set of faces or vertices is said to be in a corresponding position around Xn.

Fundamental properties of axes in poly symmetric polyhedra

Theorem 1

All axes belonging to a finite poly symmetric polyhedra pass the same point.

Below, we will prove it in several stages.

(1)Any two axes Xn, Yn of the same number of times always intersect.

(a) When n is an even number

Assuming that Xn and Yn do not intersect (Fig. 1), these two axes are parallel to each other or not coplanar.

If they are parallel, a common perpendicular line L can be drawn at any position near the polyhedron. When the two axes are not on the same plane, there is only one common perpendicular line L between them.

In any case, if the intersections of L and X and Y are A and B, respectively, the axis penetrates the polyhedron, so the length of AB should be finite and at a finite distance from the polyhedron.

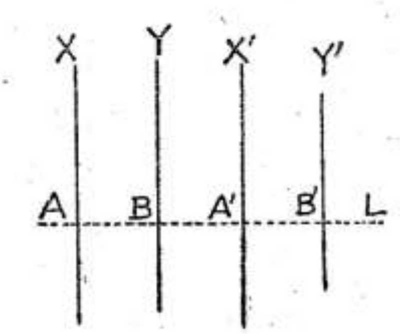

Fig. 1

Now, if you make a half turn of the polyhedron around Y, A will move to a point A' on the extension of AB, so X will pass A' and take a symmetric X' position with respect to Y. At this time, since the polyhedron overlaps with its original position, there is always one axis at the position of X'.

Similarly, when rotating the polyhedron a half-turn around X', X' is also perpendicular to L because X is perpendicular to L, so B moves to the position of B' on L and Y is It passes B' and comes to a position Y' symmetrical with respect to X'. There should be one axis here as well.

In doing so, we must admit that there are more axes perpendicular to L past points A ", B" equidistant above L. And we must admit endlessly that there are polyhedral parts near those points. This goes against the assumption that polyhedra are finite. Therefore, when n is an even number, Xn and Yn always intersect.

(b) When n is odd

In order to avoid complication, the axes are indicated by dots, and any two axes are A and A'(Fig. 2).

If the two axes do not intersect, they are either parallel to each other or not coplanar. In either case, a common perpendicular line aa' is set between the two axes (in the case of parallel, near the solid), and the intersections with A and A' are a and a', respectively. However, it is not shown in the figure.

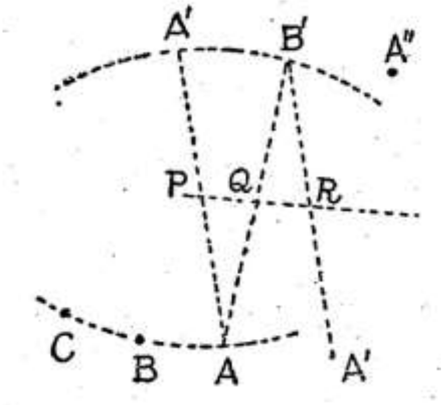

Fig.2

Now, when making one rotation of the polyhedron around A, the line segment aa' draws a circle on a plane perpendicular to A past a, and for each 1 / n rotation, a' is sequentially this circumference. It takes the positions b', c'... Above, so the axis A' is past these points and comes to a position perpendicular to the radius and equiangular to the plane of the circle. Therefore, it must be acknowledged that there are a total of n axes A', B', C'... Similarly, by rotating the polyhedron once around A', it must be acknowledged that there are n axes around it, including B, C..., etc. in the corresponding positions with A. Next, draw one of the straight lines (that is, the axes of symmetry of the two axes A and A') that pass the midpoint of the line segment aa' and are perpendicular to this and are equiangular with each of A and A' to obtain P. At this time, if he rotates the polyhedron around A by 1 / n and the position that P should take when A' overlaps B' is Q, then Q is the axis of symmetry between A and B'.

Now, when the polyhedron is rotated a half-turn around P, the axes A and A' exchange their positions with each other, even though the polyhedra do not overlap when they overlap in-situ. Therefore, all the axes whose existence should be recognized from these two axes exchange their positions by two. Therefore, within this range, P has the same properties as the bidirectional axis Z_2 as an axis.

However, if we rotate these axes a half-turn around P, B should be in the position of B', so instead of finding the existence of B' by rotating 1 / n around A, P's We must admit that B' is in the coming position of B by half a turn.

For the same reason, make a half turn around Q to find one axis A ″ at the position of A ′ . At the same time, it is recognized that A ″ is present at the upcoming position of A by half a rotation of R at the upcoming position of P. If we continue to do this, we must endlessly recognize the existence of axes whose roots are A and A'.

However, if A // A', then P // Q // R ..., and if A, A' are not on the same plane, then P, Q, R, ... are also 2 They are straight lines that are not on the same plane and do not intersect. Also, when these straight lines are arranged endlessly equidistantly on one common vertical line, the polyhedral part must exist even at an infinite distance as in (a). Such a polyhedron cannot be finite. Therefore, even when n is an odd number, Xn and Yn always intersect.

(2) Two axes Xm and Yn with different numbers always intersect.
Because, for example, when making one rotation of a polyhedron around Xm, there should be axes Yn ′ , Yn ″ , ... And if Xm // Yn, then Yn // Yn ’ , and if Xm and Yn are not coplanar, then Yn
If and Yn' do not intersect, of course, there will be two or more intersections, which is unreasonable (see 1, 3). Therefore, Xm and Yn always intersect.

(3) Any three axes Xl, Ym, Zn always pass the same point.

Since these axes intersect two by two, if these axes are not coplanar, their intersections must always be the same regardless of the values of l, m, n.

Also, if all of l, m, and n are larger than 2 even if they are all on the same plane, they are around that axis and are in the same position as one of the other two axes, and are on this plane. There must be one or more axes that are not. Therefore, even in this case, their intersections must be the same.

Next, when l = m = n = 2 and everything is on the same plane as P (Fig. 3).

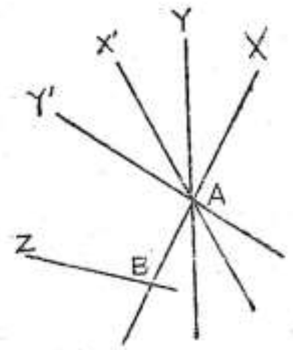

Fig.3

Let A be the intersection of X and Y, and let α be the magnitude of ∠XAY. Now, if we rotate the polyhedron a half-turn around Y, the two parts of the polyhedron separated by the plane P can exchange their positions with each other. At this time, the X' axis exists at the position where X was, and if the polyhedron is rotated half a turn around it, Y will come to the position of Y'. There should be one axis here. At this time, the two parts of the polyhedron move to the original position, and it is as if the polyhedron was rotated about 2α around A along the plane P from the initial position. Therefore, this polyhedron should have an axis Xp past A and perpendicular to the plane P. However, the value of p is one of the common divisors of α and 180 °, and is the quotient divided by 180 °.

If there is now a third axis Z and X intersects at one point B that is not A, then for the same reason as above, there is also an axis Yq perpendicular to the plane P in B, which is Xp // Yq. This is unreasonable due to (2). Therefore B must match A and X, Y, Z must pass the same point.

From the above, it can be seen that all three axes belonging to one regular polyhedron pass through the same point. Therefore, it should be inferred that all axes pass through the same point.

This proves this law.

The center of the regular polyhedron

The intersections of the axes are generally inside the polyhedron, except for the annular polyhedron. This is named the center of the poly symmetric polyhedron. And when a sphere centered on this point passes through one vertex, or one face touches the sphere, then this vertex or the vertex corresponding to the face is on the same sphere around all axes. , The surface touches the spherical surface.

Therefore, if all the vertices are co-located around any axis, this polyhedron is inscribed in one sphere. Also, if there is a similar relationship in all aspects, the polyhedron circumscribes one sphere.

Theorem 2

The number of axes belonging to one finite poly symmetric polyhedron is finite, and the angle formed by each of the two axes is also finite.

(1) If one axis passes a point in one plane, this axis is always perpendicular to that plane. This is because if they cross each other, they will never overlap in the original position during one rotation of the polyhedron around them. And if there are two or more such axes, these axes cannot intersect each other in parallel. Therefore, only one axis passes through one point in one plane.

(2) If one axis intersects one ridge, this axis is always X2, which is a two-sided symmetrical axis on both sides of this ridge that connects to it. Therefore, there is only one axis that intersects one ridge.

(3) If the axis Xn passes through one vertex, when it rotates 1 / n around it, if n> 2, then A becomes B and B becomes C in the plane around this vertex. ,... F overlaps with A. And each of these pairs of faces forms a regular polyfacet angle directly or if extended, and Xn corresponds to its axis. Also, when n = 2, A overlaps with B and B overlaps with A. In any case, only one axis passes through one vertex.

Therefore, the number of axes belonging to one regular polyhedron is finite. From this, it should be inferred that the angle formed by any two axes is also finite.

Composition of the axis of the poly symmetric polyhedron

The positional relationship between each axis is determined by the angle between them and is irrelevant to its length. Therefore, if we now make a sphere with an arbitrary radius centered on the center of the polyhedron, this sphere cuts all axes from the center to the same length. The distance on the spherical surface between the intersections represents the angle formed by the two axes passing through the point. Therefore, by considering the distribution of these points on the sphere, it is possible to clarify the positional relationship between the axes, that is, the composition of the possible axis groups. (Hereafter, the details are omitted and only the results are given to conclude this manuscript.)

Types of axis groups

(1) Regular icosahedron axis group X_5 is the main axis (refers to the axis with the maximum number of axes contained in one polyhedron, and the others are called sub-axis): 12 X_5 and 20 Y_3. It consists of 30 Z_2s, and their mutual positional relationship is the same as the axis group included in the regular dodecahedron and the regular icosahedron. That is, for each face of the icosahedron, X_5 passes through each vertex, Y_3 passes through the center of the face, and Z_2 passes through the midpoint of each side. (Fig. 4)

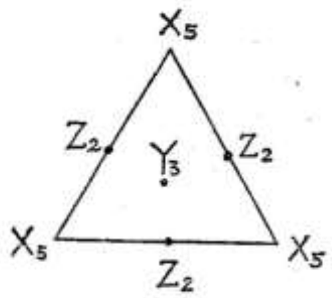

Fig.4

This axis group is called an icosahedron axis group, and the polyhedron including this is called an icosahedron system or a five-fold symmetric polyhedron. This is expressed as follows.

$$12X_5 + 20Y_3 + 30Z_2$$

(2) The polyhedron with the regular octahedron axis group X_4 as the main axis consists of the following combinations.

$$6X_4 + 8Y_3 + 12Z_2$$

It is the same as the axis group of the octahedron, where X_4 passes through the vertices of each face, Y_3 passes through its center, and Z_2 passes through the midpoint of each side. This is called a regular octahedron axis group, and a polyhedron with this is called an octahedron system or a four-fold symmetric polyhedron.

(3) A polyhedron whose main axis is the regular tetrahedron axis group

X_3 consists of the following combinations.

$$4X_3 + 4Y_3 + 6Z_2$$

It is the same as the axis group of the regular tetrahedron, where X_3 passes through the vertices of each face, Y_3 passes through its center, and Z_2 passes through the midpoint of each side. (Fig. 5) This is called a regular tetrahedron axis group, and a polyhedron having this is called a tetrahedron system or a three-fold symmetric polyhedron.

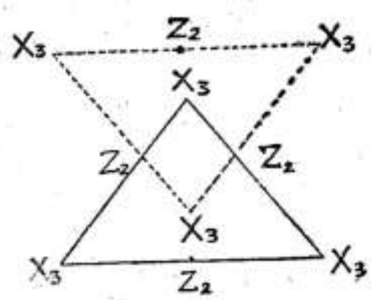

Fig.5

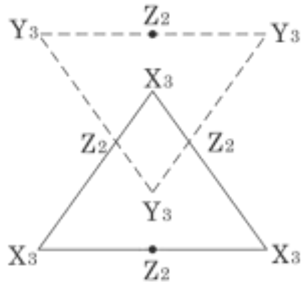

Fig.5'

In this group of axes, two X_3s and two Y_3s are in a straight line facing in the opposite direction. However, it never overlaps when changing position due to the rotation of other axes, and should be considered as a completely different kind. Therefore, instead of using X_3 as the main axis, a polyhedron having the same shape as that of the previous axis group is obtained by taking the axis group with Y_3 as the main axis (broken line in Fig. 5).

$$4Y_3 + 4X_3 + 6Z_2$$

To distinguish between them, one is named positive (+) and the other is named negative (-). For example, it is called a positive / regular tetrahedron and a negative / regular tetrahedron.

(4) Bipyramidal axis group

This group of axes has a configuration with two axes Xm forming a straight line in opposite directions as the main axis.

$$2X_m + mY_2 + mZ_2$$

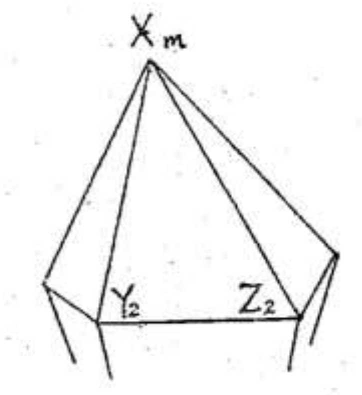

Fig.6

Xm passes through the common vertices of each triangular surface of the bipyramid, and the other two axes Y_2 and Z_2 both pass through the other two vertices. (Fig. 6)
This is called a bipyramidal axis group, and a polyhedron having this axis group is called a bipyramid system or an m-fold symmetric polyhedron.
There are no possible axis groups or polyhedra other than the above four types.

Caution

Many of these polyhedra can be found in crystals. The equiaxed crystal system of the crystal belongs to the groups (2) and (3), and the tetragonal, orthorhombic, and hexagonal systems belong to (4). Monoclinic and triclinic systems are not present in the poly symmetric polyhedra. Further, the one belonging to (1) in the poly symmetric polyhedra does not exist in the crystals.

(The end)

Composition of the axis of the poly symmetric polyhedron

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Tamekichi Hishida

In the 50th anniversary issue of the Tokyo Physical School, we proved that all the axes of poly symmetric polyhedron pass the same point and the number of axes is finite, and finally made a conclusion about the types of axis groups that can be established. .. This issue discusses the details.

*The English translation of the text below is omitted, and only the figure is shown (by translator Hiroshi).

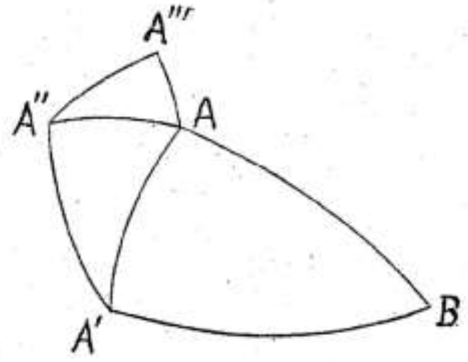


Fig.1

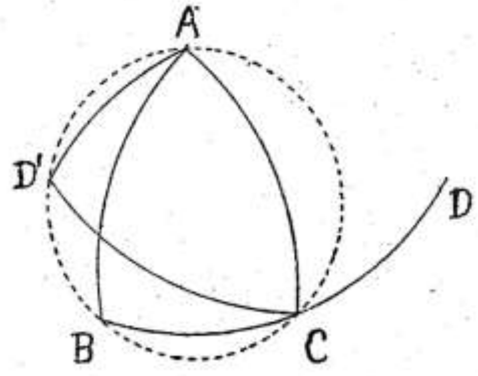


Fig.2

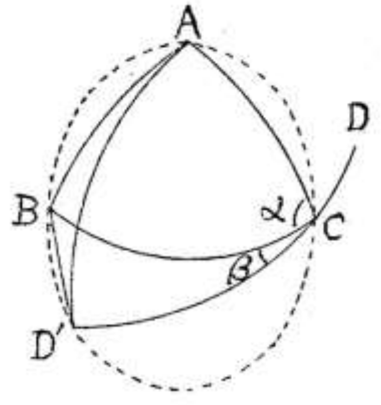


Fig.3

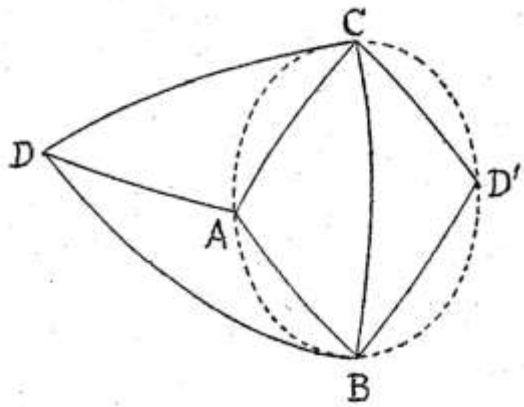


Fig.4

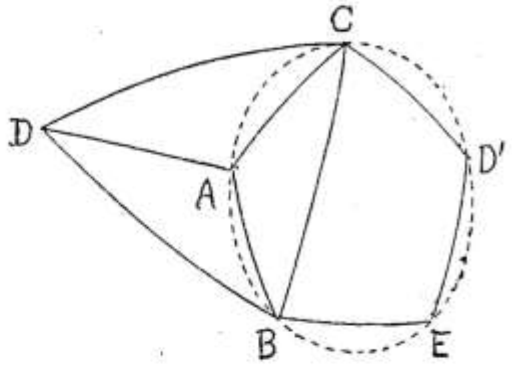
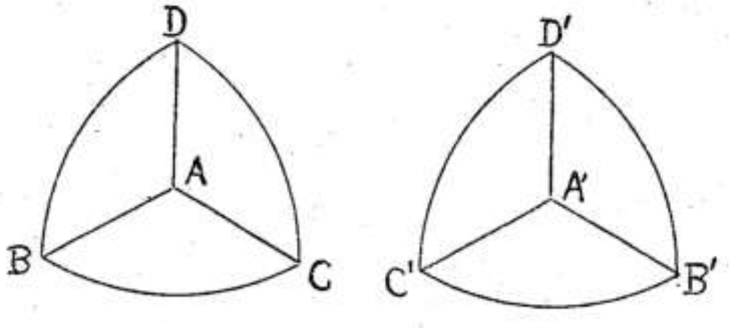


Fig.5



Fif.6

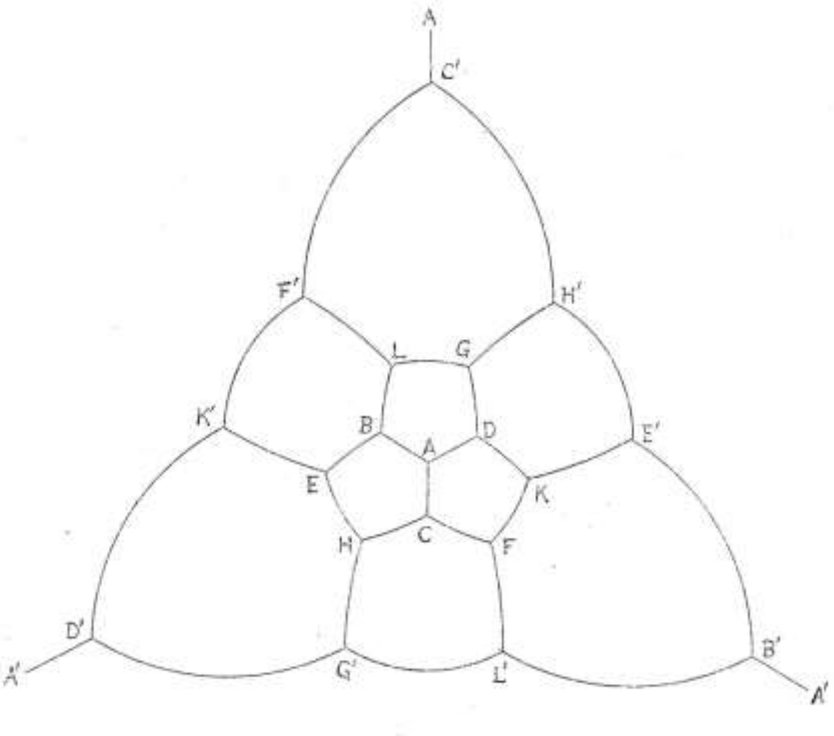
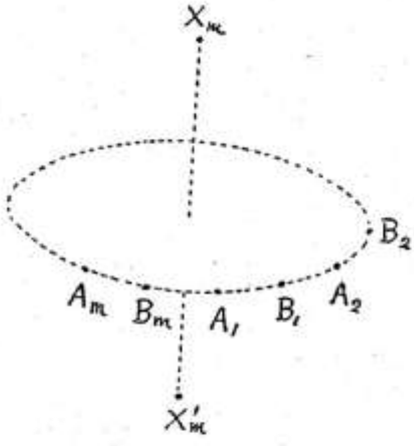


Fig.7



Fif.8

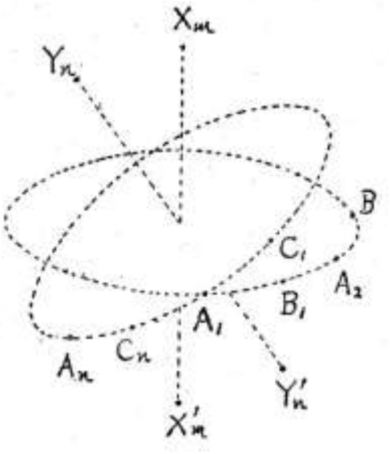
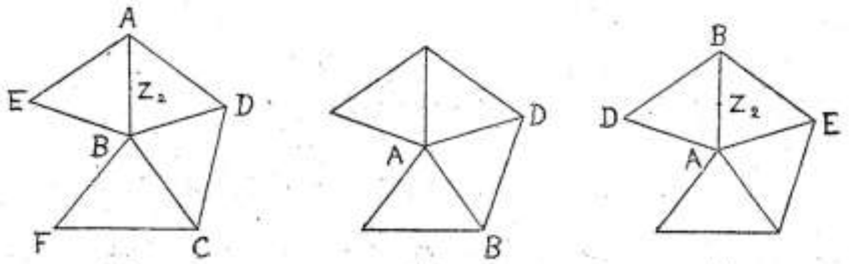


Fig.9



(a) (b) (c)

Fig.10

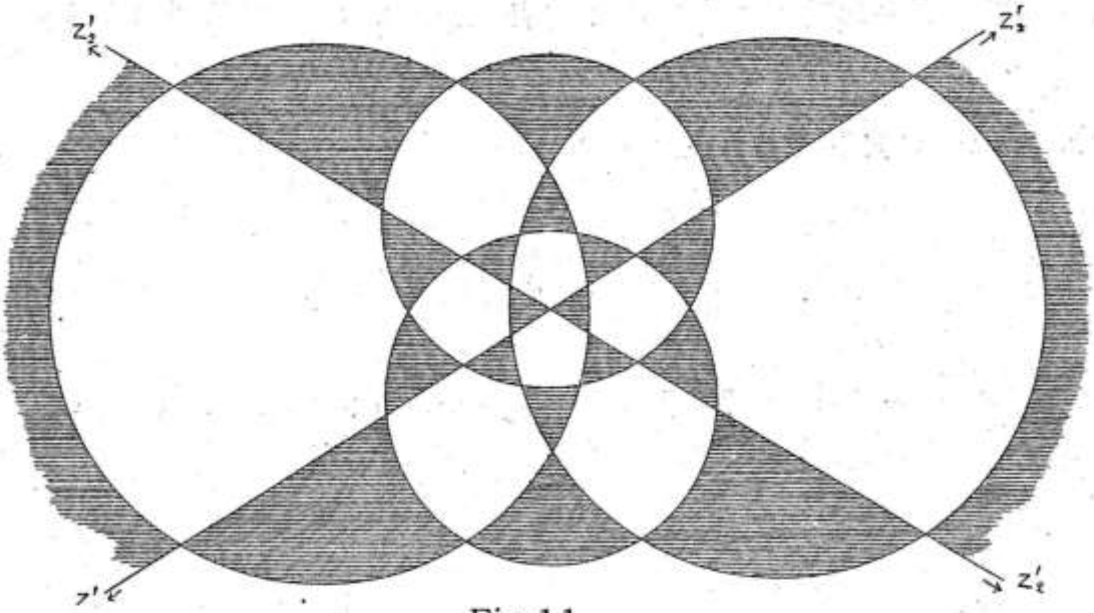


Fig.11

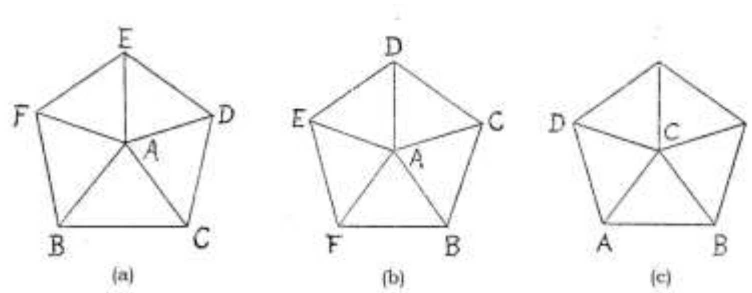


Fig. 12

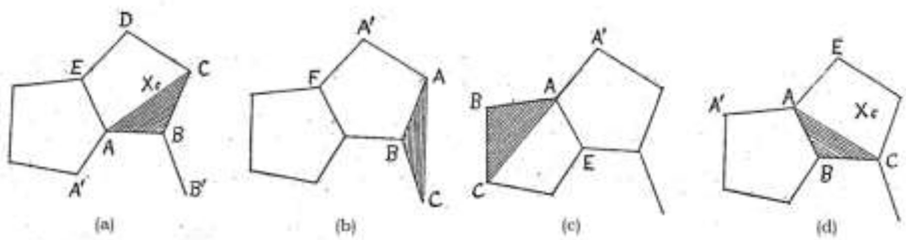


Fig. 13

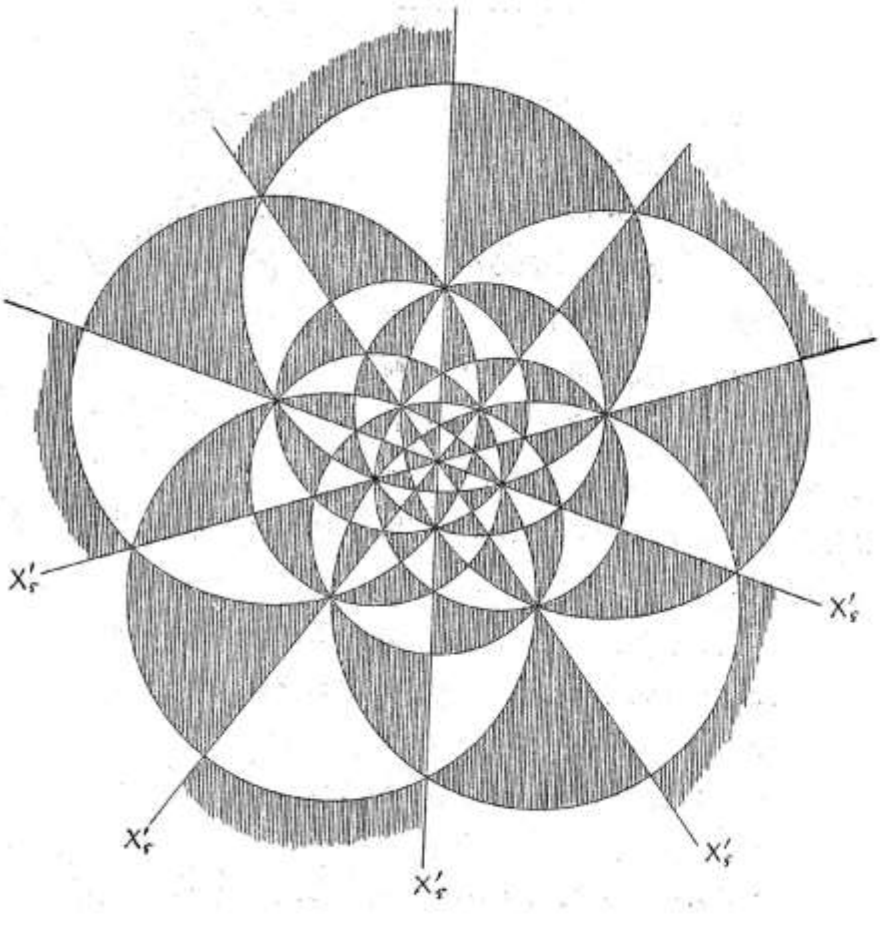


Fig. 14

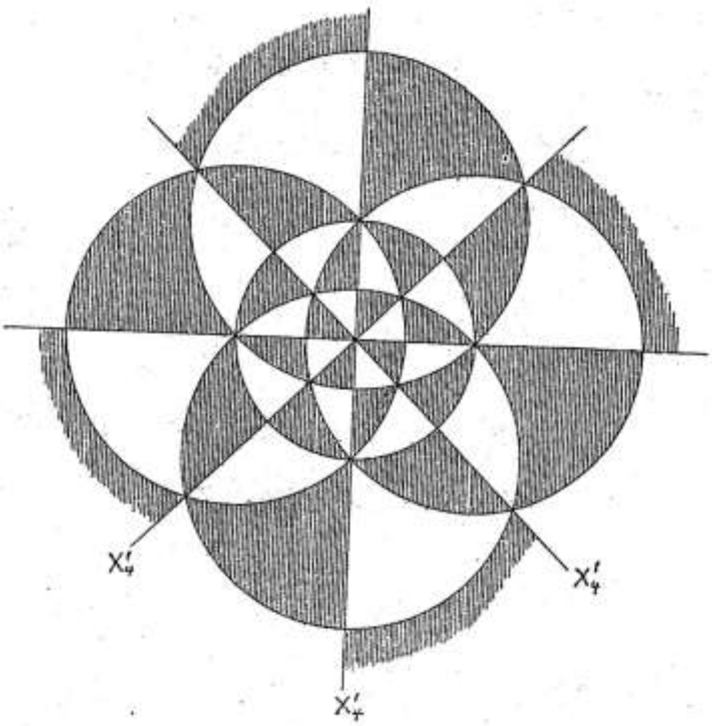


Fig.15

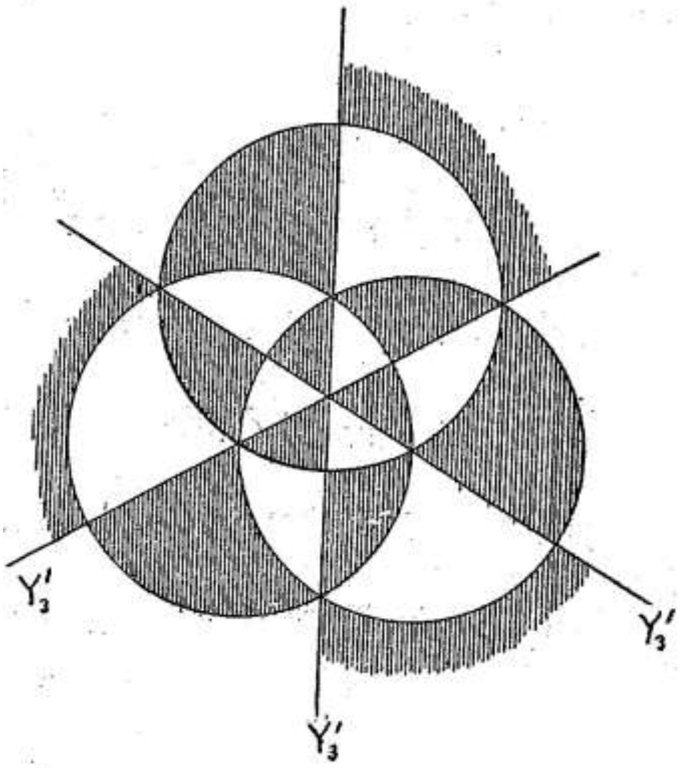


Fig.16

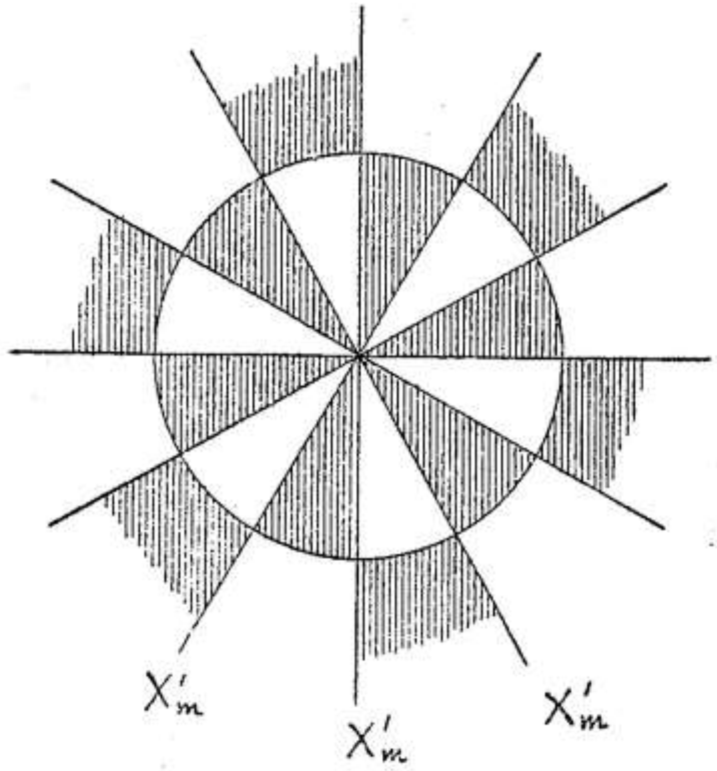


Fig. 17